Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

THE

ASSURANCE MAGAZINE,

AND

JOURNAL

OF THE

INSTITUTE OF ACTUARIES.

Solutions of General Problems in Survivorships. *By* WILLIAM MATTHEW MAKEHAM, *Fellow of the Institute of Actuaries.*

[Read before the Institute, 28th November, 1864.]

THE following collection of problems is intended to serve two purposes—first, as a continuation of a former article, published in vol. x. of this *Magazine,* under the title of "Solutions of the Compound-Survivorship Assurance Problems," which treats of cases involving three lives only; and secondly, as an introduction, or preliminary step, to a complete exposition of the writer's method of constructing mortality and annuity tables upon Mr. Gompertz's celebrated hypothesis, a brief account of which method will be found in the eighth and ninth volumes.

In the solutions of the three-life cases, it was shown that all the compound-survivorship problems, which had been approximatively solved by Baily and Milne, are reducible to the compound-survivorship annuity $\rceil_{\substack{\mathrm{BC}\\ \mathrm{I}}}\mathrm{A}$ (combined with simple-survivorship reversions), and consequently that the construction of complete tables of this function will enable us to effect the numerical solution of the problems in question. In the exposition above referred to, a method of constructing these tables will be given, by which the labour is very much reduced; but as even by this method the work of construction is by no means inconsiderable, it may not be considered useless to ascertain to what extent such tables when constructed will be serviceable—that is, what other problems they will enable us to solve in addition to those already investigated.

The conclusion derived from an examination of the following solutions is, that *all* cases involving not more than *four* lives are within our power, by the construction of complete tables of $\rceil_{\substack{BC\\I}}A$, $\rceil_{\substack{BCD\\I}}A$, and $\rceil_{\substack{CD\\I}}AB$. This is shown by the solutions of Problems IV. and XII., which determine the values of $\rceil_{\substack{\overline{B\,C\,D}\\III\,II\,I}}A$ and $\underset{IV}{\mathfrak{A}}\underset{III}{\mathbb{B}}\underset{II}{\mathbb{C}}\underset{I}{\mathbb{D}}$, where each of the three lives which are to fail before A must die in a given order. The investigation is confined to these two problems and those which lead to them, it being no part of the writer's plan to give a complete collection of the numerous cases which might be formed by combining two or more of these together.

There are three fundamental problems upon which the solution of the others depends, viz., the first, the fifth, and the ninth. The exact solution of these fundamental problems varies according to the nature of the function expressing the law of mortality—which is not the case with any of the others, except so far as they are dependent upon the fundamental problems. The solutions given of these fundamental problems are approximations only, although it is believed very close ones; but *exact* solutions of these cases will be given in treating of the law of mortality, which will enable us to test the approximations now given.

The very elegant and convenient expression for the value of the annuity $\rceil_{\substack{BC\\I}}A$, viz., $S\frac{{}^{n-1}b-{}^{n}b}{b}\cdot\frac{{}^{n-\frac{1}{2}}c}{c}\cdot\rceil^{n-1}A$ (which denotes an annuity on *A*'s life deferred until the death of *B* during *C*'s lifetime), was, it appears, first given to the world by Mr. Griffith Davies. The principle of this solution is a very important one—it has been adopted in the solution of Problem IX; a deferred survivorship assurance, on *A*'s life against *B*'s, being substituted for a deferred annuity on the life of *A*.

For the purpose of generalising the solutions, I have extended the signification of the notation used by Mr. Milne. Thus, the characters A and $\mathfrak{A}$ denote annuities and assurances on the joint existence of any given number of lives, A_1, A_2, A_3, . . . while a denotes the product of the numbers living at the several ages of these lives. The characters ${}^{\frac{1}{2}}A$, ${}^{\frac{1}{2}}\mathfrak{A}$, ${}^{\frac{1}{2}}a$, and $A_{\frac{1}{2}}$, $\mathfrak{A}_{\frac{1}{2}}$, $a_{\frac{1}{2}}$, signify the same on lives each a half year older and younger respectively. The adoption of Mr. Milne's system of notation has prevented the use of the "commutation" method, by which many of the formulæ would have been rendered much more concise and elegant, but anyone who may take the trouble to peruse this article will see where that method may be substituted with advantage.

PROBLEM I.

To determine ($\rceil^{\mathrm{BC}}_{\mathrm{I}}\mathrm{A}$) the present value of an annuity on the joint lives A_1, A_2, A_3, . . . after the failure of the joint existence of the lives B_1, B_2, B_3, provided that event shall happen during the joint existence of the lives C_1, C_2, C_3.

Solution.

The contingent benefit, of which the value is to be determined, is, in fact, a deferred annuity on the joint lives (A), the first payment of which is not to be made until the end of the year in which the joint lives (B) shall fail during the joint existence of the lives (C). If we have tables showing the number of the combinations (similar to each of these sets) which survive each year of age, and the decrements thereof, we may approximate with sufficient accuracy to the chance of the joint lives (C) surviving the joint lives (B) in the nth year by the formula $\frac{{}^{n-1}b-{}^{n}b}{b}\cdot\frac{{}^{n-1}c+{}^{n}c}{2c}$. In the case of a single life in each set, this expression is exact upon the hypothesis of a uniform distribution of the deaths of each year; but in cases involving a plurality of lives in each set, it will be found abundantly accurate for all practical purposes.

The value of an annuity on the joint lives (A) deferred $n-1$ years, or $\rceil_{n-1}\mathrm{A}$, is $\frac{{}^{n-1}a}{a}\cdot v^{n-1}.{}^{n-1}\mathrm{A}$. Multiplying this expression into the former, we find the value of the contingency due to the nth year to be $\frac{{}^{n-1}b-{}^{n}b}{b}\cdot\frac{{}^{n-1}c+{}^{n}c}{2c}\cdot\frac{{}^{n-1}av^{n-1}}{a}\cdot{}^{n-1}\mathrm{A}$, and using S as the symbol of summation, commencing with $n=1$, we have

$$\rceil^{\mathrm{BC}}_{\mathrm{I}}\mathrm{A}=S\,\frac{{}^{n-1}b-{}^{n}b}{b}\cdot\frac{{}^{n-1}c+{}^{n}c}{2c}\cdot\frac{{}^{n-1}av^{n-1}}{a}\cdot{}^{n-1}\mathrm{A}\quad .\ .\ .\quad (1)$$

When applied to a single life in each set, this formula is in substance the same as that given by Mr. Griffith Davies, at p. 356 of his work on life contingencies.

Adapting the formula to the case in which the number of lives in each set is the same as before, but each life one year older, we have

$$\rceil^{{}^1\mathrm{B}{}^1\mathrm{C}}_{\mathrm{I}}{}^1\mathrm{A}=S\frac{{}^nb-{}^{n+1}b}{{}^1b}\cdot\frac{{}^nc+{}^{n+1}c}{2{}^1c}\cdot\frac{{}^nav^{n-1}}{{}^1a}\cdot{}^n\mathrm{A}$$

$$=\frac{(abc)}{{}^1(abc)v}\cdot S\frac{{}^nb-{}^{n+1}b}{b}\cdot\frac{{}^nc+{}^{n+1}c}{2c}\cdot\frac{{}^nav^n}{a}\cdot{}^n\mathrm{A},$$

where the expression affected by the symbol of summation is evidently $\rceil^{\mathrm{BC}}_{\mathrm{I}}\mathrm{A}$ minus its first term $\frac{1}{2}\left(1-\frac{{}^1b}{b}\right)\left(1+\frac{{}^1c}{c}\right)\mathrm{A}$. Therefore we have

$$\frac{{}^1(abc)v}{(abc)}\cdot\rceil^{{}^1\mathrm{B}{}^1\mathrm{C}}_{\mathrm{I}}{}^1\mathrm{A}=\rceil^{\mathrm{BC}}_{\mathrm{I}}\mathrm{A}-\tfrac{1}{2}\left(1-\frac{{}^1b}{b}\right)\left(1+\frac{{}^1c}{c}\right)\mathrm{A},$$

and

$$\rceil^{\mathrm{BC}}_{\mathrm{I}}\mathrm{A}=\tfrac{1}{2}\left(1-\frac{{}^1b}{b}\right)\left(1+\frac{{}^1c}{c}\right)\mathrm{A}+\frac{{}^1(abc)v}{(abc)}\cdot\rceil^{{}^1\mathrm{B}{}^1\mathrm{C}}_{\mathrm{I}}{}^1\mathrm{A}\quad .\ .\quad (2),$$

by which formula the value of the annuity $\rceil^{\mathrm{BC}}_{\mathrm{I}}\mathrm{A}$ may be derived from the value of a similar annuity where all the lives involved are one year older.

Returning to the formula (1) we get, by multiplying out,

$$\rceil^{\mathrm{BC}}_{\mathrm{I}}\mathrm{A}=S.\frac{{}^{n-1}(bc)-{}^n(bc)+{}^{n-1}b.{}^nc-{}^nb.{}^{n-1}c}{2(bc)}\cdot\frac{{}^{n-1}a.v^{n-1}}{a}\cdot{}^{n-1}\mathrm{A}$$

$$=\tfrac{1}{2}S.\left(\frac{{}^{n-1}(bc)-{}^n(bc)}{(bc)}-\frac{{}^{n-1}({}^1bc)}{({}^1bc)}\cdot\frac{{}^1b}{b}+\frac{{}^{n-1}(b{}^1c)}{(b{}^1c)}\cdot\frac{{}^1c}{c}\right)\cdot\frac{{}^{n-1}a.v^{n-1}}{a}\cdot{}^{n-1}\mathrm{A}$$

$$=\tfrac{1}{2}\left[S.\frac{{}^{n-1}(bc)-{}^n(bc)}{(bc)}\cdot\frac{{}^{n-1}a.v^{n-1}}{a}\cdot{}^{n-1}\mathrm{A}-\right.$$

$$\left.S\left(\frac{{}^{n-1}({}^1bc)}{({}^1bc)}\cdot\frac{{}^1b}{b}-\frac{{}^{n-1}(b{}^1c)}{(b{}^1c)}\cdot\frac{{}^1c}{c}\right)\cdot\frac{{}^{n-1}a.v^{n-1}}{a}\cdot{}^{n-1}\mathrm{A}\right].$$

The first term within the brackets [ ] denotes the value of a deferred annuity on the joint lives (*A*), the first payment of which is to be made at the end of the year in which the joint lives (*BC*) shall fail; in other words, a reversionary annuity on the joint lives (*A*) after the joint lives (*BC*), the value of which may be shown to be A—ABC. The remaining terms resolve themselves into the values of annuities of a novel and somewhat remarkable character, the form of which is

$$S\cdot\frac{{}^{n-1}(bc)}{(bc)}\cdot\frac{{}^{n-1}a.v^{n-1}}{a}\cdot{}^{n-1}\mathrm{A},$$

or

$$\mathrm{A}+\frac{{}^1(bc)}{(bc)}\cdot\rceil_{1}\mathrm{A}+\frac{{}^2(bc)}{(bc)}\rceil_{2}\mathrm{A}+\frac{{}^3(bc)}{(bc)}\rceil_{3}\mathrm{A}+\ .\ .\ .\ .$$

which it may readily be perceived is the value of an annuity on the joint lives *A*, beginning with £1, and increasing by £1 every year during the joint existence of the lives (*BC*). Denoting an annuity of this description by A^{BC}, and making the requisite substitutions in the last equation, we have

$$\urcorner_{\mathrm{I}}^{\mathrm{BC}}\mathrm{A}=\tfrac{1}{2}\left(\mathrm{A}-\mathrm{ABC}-\frac{{}^{1}b}{b}\cdot\mathrm{A}^{{}^{1}\mathrm{BC}}+\frac{{}^{1}c}{c}\,\mathrm{A}^{\mathrm{B}^{1}\mathrm{C}}\right)\quad .\ .\ .\ .\quad (3).$$

As this is one of the fundamental problems, upon which the solutions of the others depend, I will add two other modes of solving it.

The probability of the survivance of the joint lives (*C*) over the joint lives (*B*) taking place in the *n*th year is

$$\frac{{}^{n-1}b-{}^{n}b}{b}\cdot\frac{{}^{n-1}c+{}^{n}c}{2c},\ \text{or}\ \frac{{}^{n-1}b-{}^{n}b}{b}\cdot\frac{{}^{n-\frac{1}{2}}c}{c}.$$

But
$$\frac{{}^{n-1}b-{}^{n}b}{b}\cdot\frac{{}^{n-\frac{1}{2}}c}{c}=\frac{{}^{n-1}b^{n-\frac{1}{2}}c-{}^{n}b^{n-\frac{1}{2}}c}{bc}=\frac{{}^{n-1}(b^{\frac{1}{2}}c)-{}^{n-1}({}^{1}b^{\frac{1}{2}}c)}{bc}$$
$$=\frac{{}^{\frac{1}{2}}c}{c}\left(\frac{{}^{n-1}(b^{\frac{1}{2}}c)}{(b^{\frac{1}{2}}c)}-\frac{{}^{n-1}({}^{1}b^{\frac{1}{2}}c)}{({}^{1}b^{\frac{1}{2}}c)}\frac{{}^{1}b}{b}\right).$$

Multiply by $\urcorner_{n-1}\mathrm{A}$, and we get

$$\urcorner_{\mathrm{I}}^{\mathrm{BC}}\mathrm{A}=\frac{{}^{\frac{1}{2}}c}{c}\left(S\cdot\frac{{}^{n-1}(b^{\frac{1}{2}}c)}{(b^{\frac{1}{2}}c)}\urcorner_{n-1}\mathrm{A}-S\cdot\frac{{}^{n-1}({}^{1}b^{\frac{1}{2}}c)}{({}^{1}b^{\frac{1}{2}}c)}\urcorner_{n-1}\mathrm{A}\cdot\frac{{}^{1}b}{b}\right)$$
$$=\frac{{}^{\frac{1}{2}}c}{c}\left(\mathrm{A}^{\mathrm{B}^{\frac{1}{2}}\mathrm{C}}-\mathrm{A}^{{}^{1}\mathrm{B}^{\frac{1}{2}}\mathrm{C}}\cdot\frac{{}^{1}b}{b}\right)\quad .\ .\ .\ .\ .\ .\quad (4).$$

Again, as the first payment of the annuity $\urcorner_{\mathrm{I}}^{\mathrm{BC}}\mathrm{A}$ is to be made at the end of the year in which the survivance of the joint lives (*C*) over the joint lives (*B*) takes place, we may suppose it to fall due, upon an average, six months after that event. But an annuity has the same value, whether it be payable annually, half-yearly, or quarterly, or, indeed, at any other interval, if the next payment falls due at the expiration of one-half of the period intervening between the payments; and its value in that case may always be expressed by that of an annuity commencing immediately and payable momently.* Denoting, then, a momently annuity on the joint lives (*A*) by $\underset{\circ}{\mathrm{A}}$, it appears that

---

* It is, I believe, sometimes the practice, in making the periodical valuations of an Assurance Office, to substitute for *half-yearly* and *quarterly* premiums the corresponding *annual* payments. But as these half-yearly and quarterly policies, one with another, will become renewable at the expiration of one-half of the interval of renewal, it follows, from the circumstance above mentioned, that this substitution is not only unnecessary, but absolutely incorrect. The proper course is, to take the sum of the payments due in the year on each policy, whether one, two, or four, and value them as if they were payable altogether at the expiration of half a year. Of course, these remarks do not apply when the exact period at which the next premium becomes due, on each policy, is taken into account. In that case, the substitution referred to would be quite correct.

$$\urcorner^{BC}_{\mathrm{I}}A=\urcorner^{BC}_{\mathrm{I}}\underset{\circ}{A}=S\cdot\frac{{}^{n-1}b-{}^{n}b}{b}\cdot\frac{{}^{n-\frac{1}{2}}c}{c}\cdot\urcorner_{n-\frac{1}{2}}\underset{\circ}{A}.$$

That is, an annuity, the first payment of which is to be made at the end of the year in which the survivorship takes place, and to be paid annually thereafter, may be considered equivalent to a momently annuity commencing at the instant of survivorship.

Substituting in the last expression ${}^{n-\frac{1}{2}}(b_{\frac{1}{2}})$ for ${}^{n-1}b$, and ${}^{n-\frac{1}{2}}({}^{\frac{1}{2}}b)$ for ${}^{n}b$, we have

$$\urcorner^{BC}_{\mathrm{I}}A=\frac{b_{\frac{1}{2}}}{b}S\frac{{}^{n-\frac{1}{2}}(b_{\frac{1}{2}}c)}{(b_{\frac{1}{2}}c)}\urcorner_{n-\frac{1}{2}}\underset{\circ}{A}-\frac{{}^{\frac{1}{2}}b}{b}S\frac{{}^{n-\frac{1}{2}}({}^{\frac{1}{2}}bc)}{({}^{\frac{1}{2}}bc)}\urcorner_{n-\frac{1}{2}}\underset{\circ}{A}.$$

The expression following the symbol of summation may be considered as a momently annuity on the joint lives (A), commencing with nothing, but increasing uniformly, by infinitely small increments at the rate of £1 per annum during the joint lives ($B_{\frac{1}{2}}C$) or (${}^{\frac{1}{2}}BC$). Denoting this by $\underset{\circ}{A}^{B_{\frac{1}{2}}C}$ or $\underset{\circ}{A}^{\frac{1}{2}BC}$ we have

$$\urcorner^{BC}_{\mathrm{I}}A=\frac{1}{b}\left(\underset{\circ}{A}^{B_{\frac{1}{2}}C}\,b_{\frac{1}{2}}-\underset{\circ}{A}^{\frac{1}{2}BC}\,{}^{\frac{1}{2}}b\right)\quad.\quad.\quad.\quad.\quad.\quad(5).$$

PROBLEM II.

To determine ($\urcorner^{\overline{BC}}_{\mathrm{II\,I}}A$) the present value of an annuity on the joint lives A_1, A_2, A_3, . . . after the failure of the joint lives B_1, B_2, B_3, . . . provided that event shall be preceded by the failure of the joint lives C_1, C_2, C_3, . . .

Solution.

$$\urcorner^{BC}_{\mathrm{I}}A+\urcorner^{\overline{BC}}_{\mathrm{II\,I}}A=\urcorner_{B}A$$
$$=A-AB$$
$$\therefore\ \urcorner^{\overline{BC}}_{\mathrm{II\,I}}A=A-AB-\urcorner^{BC}_{\mathrm{I}}A.$$

PROBLEM III.

To determine ($\urcorner_{\underset{\mathrm{II.III}}{B}\,\underset{\mathrm{I}}{C}\,D}A$) the value of an annuity on the joint lives A_1, A_2, A_3, . . . after the failure of the joint lives B_1, B_2, B_3, . . . provided that event shall be preceded by the failure of the joint lives C_1, C_2, C_3, . . . during the existence of the joint lives D_1, D_2, D_3, . . .

Solution.

The contingent benefit stated in the problem is in effect a deferred annuity on the joint lives A, dependent upon the failure of the joint lives C during the existence of the joint lives BD, and then further deferred until the failure of the joint lives B. If to this annuity there be added an annuity on the joint lives A, commencing immediately on the failure of the joint lives C during the existence of the joint lives BD, but ceasing on the failure of the joint lives B, the result will be an annuity on the joint lives A after the failure of the joint lives C during the existence of the joint lives BD. That is to say—

$$\urcorner_{\substack{\mathrm{B\ CD}\\ \mathrm{II.III\ I}}}\mathrm{A}+\urcorner_{\substack{\mathrm{CD}\\ \mathrm{I}}}\mathrm{AB}=\urcorner_{\substack{\mathrm{CBD}\\ \mathrm{I}}}\mathrm{A}$$

$$\therefore \urcorner_{\substack{\mathrm{B\ CD}\\ \mathrm{II.III\ I}}}\mathrm{A}=\urcorner_{\substack{\mathrm{CBD}\\ \mathrm{I}}}\mathrm{A}-\urcorner_{\substack{\mathrm{CD}\\ \mathrm{I}}}\mathrm{AB}.$$

The two terms of the second member of the last equation are resolved by Problem I.

PROBLEM IV.

To determine $(\urcorner_{\substack{\overline{\mathrm{B\,C\,D}}\\ \mathrm{III\ II\ I}}}\mathrm{A})$ the value of an annuity on the joint lives A_1, A_2, A_3, . . . after the failure of the joint lives B_1, B_2, B_3, . . . provided that event shall be preceded by the failure of the joint lives C_1, C_2, C_3, . . . after the failure of the joint lives D_1, D_2, D_3, . . .

Solution.

$$\urcorner_{\substack{\overline{\mathrm{B\,C\,D}}\\ \mathrm{III\ II\ I}}}\mathrm{A}+\urcorner_{\substack{\mathrm{B\ \ C\ \ D}\\ \mathrm{II.III\ I}}}\mathrm{A}=\urcorner_{\substack{\overline{\mathrm{BC}}\\ \mathrm{II\ I}}}\mathrm{A}=\mathrm{A}-\mathrm{AB}-\urcorner_{\substack{\mathrm{BC}\\ \mathrm{I}}}\mathrm{A}\ \text{(Prob. II.)}$$

$$\therefore \urcorner_{\substack{\overline{\mathrm{B\,C\,D}}\\ \mathrm{III\ II\ I}}}\mathrm{A}=\mathrm{A}-\mathrm{AB}-\urcorner_{\substack{\mathrm{BC}\\ \mathrm{I}}}\mathrm{A}-\urcorner_{\substack{\mathrm{B\ \ C\ \ D}\\ \mathrm{II.III\ I}}}\mathrm{A}$$

$$=\mathrm{A}-\mathrm{AB}-\urcorner_{\substack{\mathrm{BC}\\ \mathrm{I}}}\mathrm{A}-\urcorner_{\substack{\mathrm{CBD}\\ \mathrm{I}}}\mathrm{A}+\urcorner_{\substack{\mathrm{CD}\\ \mathrm{I}}}\mathrm{AB}.$$

PROBLEM V.

To determine $(\underset{\mathrm{I}}{\mathfrak{A}}\mathbb{B})$ the present value of £1 payable upon the ailure of the joint lives A_1, A_2, A_3, . . . during the existence of the joint lives B_1, B_2, B_3, . . .

Solution.

Adopting the approximative formula used in the solution of Problem I. for the chance of the joint lives (A) failing before the joint lives (B) during the nth year, we shall have

$$
\begin{aligned}
\underset{\mathrm{I}}{\mathfrak{A}}\mathbb{B} &= S\cdot\frac{{}^{n-1}a-{}^{n}a}{a}\cdot\frac{{}^{n-1}b+{}^{n}b}{2b}\cdot v^{n}\\
&= S\cdot\frac{{}^{n-1}(ab)-{}^{n}(ab)+{}^{n-1}a\,{}^{n}b-{}^{n}a\,{}^{n-1}b}{2(ab)}\cdot v^{n}\\
&= \tfrac{1}{2}S\left(\frac{{}^{n-1}(ab)-{}^{n}(ab)}{(ab)}\cdot v^{n}+\frac{{}^{n}(a_1b)}{(a_1b)}\cdot v^{n}\cdot\frac{a_1}{a}-\frac{{}^{n}(ab_1)}{(ab_1)}\cdot v^{n}\cdot\frac{b_1}{b}\right)\\
&= \tfrac{1}{2}\left(\mathfrak{A}\mathfrak{B}+\frac{a_1}{a}\cdot \mathrm{A}_1\mathrm{B}-\frac{b_1}{b}\mathrm{AB}_1\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).
\end{aligned}
$$

When there is one life only in each set, this formula is exact, upon the hypothesis of a uniform distribution of the deaths of each year, and is identical with the formula of Mr. Milne. In other cases, however, it will be found sufficiently accurate for all practical purposes.

Again, $({}^{n-1}a-{}^{n}a).{}^{n-\frac{1}{2}}b.v^{n}={}^{n}(a_1b_{\frac{1}{2}})v^{n}-{}^{n}(ab_{\frac{1}{2}})v^{n}$, whence

$$
\begin{aligned}
\underset{\mathrm{I}}{\mathfrak{A}}\mathbb{B} &= S\frac{{}^{n}(a_1b_{\frac{1}{2}})v^{n}}{(ab)}-S\frac{{}^{n}(ab_{\frac{1}{2}})v^{n}}{(ab)}\\
&= \left(\mathrm{A}_1\mathrm{B}_{\frac{1}{2}}\cdot\frac{a_1}{a}-\mathrm{AB}_{\frac{1}{2}}\right)\frac{b_{\frac{1}{2}}}{b} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).^{*}
\end{aligned}
$$

Lastly, $({}^{n-1}a-{}^{n}a)^{n-\frac{1}{2}}b.v^{n}=\{{}^{n-\frac{1}{2}}(a_{\frac{1}{2}}b).v^{n-\frac{1}{2}}-{}^{n-\frac{1}{2}}({}^{\frac{1}{2}}ab)v^{n-\frac{1}{2}}\}v^{\frac{1}{2}}$

$$
\begin{aligned}
\underset{\mathrm{I}}{\mathfrak{A}}\mathbb{B} &= \frac{v^{\frac{1}{2}}}{a}\left(S\frac{{}^{n-\frac{1}{2}}(a_{\frac{1}{2}}b)v^{n-\frac{1}{2}}}{(a_{\frac{1}{2}}b)}a_{\frac{1}{2}}-S\frac{{}^{n-\frac{1}{2}}({}^{\frac{1}{2}}ab)v^{n-\frac{1}{2}}}{({}^{\frac{1}{2}}ab)}\cdot{}^{\frac{1}{2}}a\right)\\
&= \frac{v^{\frac{1}{2}}}{a}\left(\underbrace{\mathrm{A}_{\frac{1}{2}}\mathrm{B}}_{\circ}.\,a_{\frac{1}{2}}-{}^{\frac{1}{2}}\underbrace{\mathrm{AB}}_{\circ}.{}^{\frac{1}{2}}a\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),
\end{aligned}
$$

the expression $\underbrace{\mathrm{AB}}_{\circ}$ denoting the value of an annuity payable momently on the joint lives (AB). If we omit the factor $v^{\frac{1}{2}}$, this formula will give the value of the assurance payable at the instant of death.

PROBLEM VI.

To determine ($\underset{\mathrm{II}}{\mathfrak{A}}\mathbb{B}$) the present value of £1 payable upon the failure of the joint lives A_1, A_2, A_3, . . . provided the joint lives B_1, B_2, B_3, . . . shall have failed previously.

Solution.

$$
\underset{\mathrm{I}}{\mathfrak{A}}\mathbb{B}+\underset{\mathrm{II}}{\mathfrak{A}}\mathbb{B}=\mathfrak{A}
$$

$$
\therefore\ \underset{\mathrm{II}}{\mathfrak{A}}\mathbb{B}=\mathfrak{A}-\underset{\mathrm{I}}{\mathfrak{A}}\mathbb{B}.
$$

---

* This formula is given by Mr. Gompertz, in his *Analysis Applicable to the Estimation of Life Contingencies.*

PROBLEM VII.

To determine $(\underset{\text{II.III}}{\mathfrak{A}}\mathbb{B}\underset{\text{I}}{\mathbb{C}})$ the present value of £1 payable on the failure of the joint lives A_1, A_2, A_3, . . . provided the joint lives C_1, C_2, C_3, . . . shall have failed previously during the existence of the joint lives B_1, B_2, B_3, . . .

Solution.

$$\rceil_{\mathrm{B}\underset{\mathrm{I}}{\mathrm{C}}}\mathrm{A}+\underset{\text{II.III}}{\mathfrak{A}}\mathbb{B}\underset{\text{I}}{\mathbb{C}}\left(1+\frac{1}{r}\right)=\underset{\text{I}}{\mathfrak{C}}\mathbb{A}\mathbb{B}\left(1+\frac{1}{r}\right)$$

$$\underset{\text{II.III}}{\mathfrak{A}}\mathbb{B}\underset{\text{I}}{\mathbb{C}}=\underset{\text{I}}{\mathfrak{C}}\mathbb{A}\mathbb{B}-\rceil_{\mathrm{B}\underset{\mathrm{I}}{\mathrm{C}}}\mathrm{A}(1-v).$$

PROBLEM VIII.

To determine $(\underset{\text{III}}{\mathfrak{A}}\mathbb{B}\underset{\text{II}}{\mathbb{C}})$ the present value of £1 payable on the failure of the joint lives A_1, A_2, A_3, . . . provided the joint lives C_1, C_2, C_3, . . . shall have failed previously after the failure of the joint lives B_1, B_2, B_3, ...

Solution.

$$\underset{\text{III}}{\mathfrak{A}}\mathbb{B}\underset{\text{II}}{\mathbb{C}}+\underset{\text{II.III}}{\mathfrak{A}}\mathbb{B}\underset{\text{I}}{\mathbb{C}}=\underset{\text{II}}{\mathfrak{A}}\mathbb{C}=\mathfrak{A}-\underset{\text{I}}{\mathfrak{A}}\mathbb{C}$$

$$\begin{aligned}\underset{\text{III}}{\mathfrak{A}}\mathbb{B}\underset{\text{II}}{\mathbb{C}}&=\mathfrak{A}-\underset{\text{I}}{\mathfrak{A}}\mathbb{C}-\underset{\text{II.III}}{\mathfrak{A}}\mathbb{B}\underset{\text{I}}{\mathbb{C}}\\&=\mathfrak{A}-\underset{\text{I}}{\mathfrak{A}}\mathbb{C}-\underset{\text{I}}{\mathfrak{C}}\mathbb{A}\mathbb{B}+\rceil_{\mathrm{B}\underset{\mathrm{I}}{\mathrm{C}}}\mathrm{A}(1-v).\end{aligned}$$

PROBLEM IX.

To determine $(\underset{\substack{\text{II.III}\\ \text{c}\\ \text{II}}}{\mathfrak{A}}\mathbb{B}\mathbb{C}\underset{\text{I}}{\mathbb{D}})$ the present value of £1 payable on the failure of the joint lives A_1, A_2, A_3, . . . provided that event shall occur after the failure of the joint lives D_1, D_2, D_3, . . . and before the failure of the joint lives B_1, B_2, B_3 . . .; and provided the joint lives C_1, C_2, C_3, . . . shall survive the joint lives D_1, D_2, D_3, . . .

Solution.

The contingent benefit stated in this problem, is, in effect, an assurance on the joint lives A_1, A_2, A_3, . . . against the joint lives B_1, B_2, B_3 . . . deferred until the failure of the joint lives D_1, D_2, D_3, . . . during the existence of the joint lives C_1, C_2, C_3. . . . The value of the contingency due to the nth year will therefore be $\frac{{}^{n-1}d-{}^{n}d}{d}\cdot\frac{{}^{n-1}c+{}^{n}c}{2c}\,\rceil_{n-\frac{1}{2}}\underset{\text{I}}{\mathfrak{A}}\mathbb{B}$, on the assumption that the survivance of the joint lives (C) over the joint lives (D), if occurring in the *n*th year, will take place in the middle of that year. We have, therefore,

$$\underset{\substack{\mathrm{ii.iii}\\ \mathrm{c}\\ \mathrm{ii}}}{\mathfrak{A}}\mathbb{B}\mathbb{C}\underset{\mathrm{i}}{\mathbb{D}}=S\cdot\frac{{}^{n-1}d-{}^{n}d}{d}\cdot\frac{{}^{n-1}c+{}^{n}c}{2c}\,\rceil_{n-\frac{1}{2}}\underset{\mathrm{i}}{\mathfrak{A}}\mathbb{B}\quad .\ .\ .\ .\quad (1)$$

Now, referring to the third solution of Problem V., we find that

$$\underset{\mathrm{i}}{\mathfrak{A}}\mathbb{B}=\frac{v^{\frac{1}{2}}}{a}\left(S\cdot\frac{{}^{n-\frac{1}{2}}(a_{\frac{1}{2}}b)v^{n-\frac{1}{2}}}{a_{\frac{1}{2}}b}\cdot a_{\frac{1}{2}}-S\cdot\frac{{}^{n-\frac{1}{2}}({}^{\frac{1}{2}}ab)v^{n-\frac{1}{2}}}{{}^{\frac{1}{2}}ab}\cdot{}^{\frac{1}{2}}a\right).$$

But as the assurance is to be deferred $n-\frac{1}{2}$ years, the summation, instead of commencing with $n=1$, will begin with $n-\frac{1}{2}+1$, or $n+\frac{1}{2}$; that is, the first terms of the variable quantities in the last expression will be

$$\frac{{}^{n}(a_{\frac{1}{2}}b)v^{n}}{a_{\frac{1}{2}}b}\text{ and }\frac{{}^{n}({}^{\frac{1}{2}}ab)v^{n}}{{}^{\frac{1}{2}}ab}.$$

The result will evidently be in each case an annuity deferred $n-1$ years.

We have, therefore,

$$\rceil_{n-\frac{1}{2}}\underset{\mathrm{i}}{\mathfrak{A}}\mathbb{B}=\frac{v^{\frac{1}{2}}}{a}\left(\rceil_{n-1}\mathrm{A}_{\frac{1}{2}}\mathrm{B}.a_{\frac{1}{2}}-\rceil_{n-1}{}^{\frac{1}{2}}\mathrm{AB}.{}^{\frac{1}{2}}a\right),$$

and substituting this value in (1), it appears that

$$\underset{\substack{\mathrm{ii.iii}\\ \mathrm{c}\\ \mathrm{ii}}}{\mathfrak{A}}\mathbb{B}\mathbb{C}\underset{\mathrm{i}}{\mathbb{D}}=\frac{v^{\frac{1}{2}}}{a}\left(S\frac{{}^{n-1}d-{}^{n}d}{d}\cdot\frac{{}^{n-1}c+{}^{n}c}{2c}\rceil_{n-1}\mathrm{A}_{\frac{1}{2}}\mathrm{B}.a_{\frac{1}{2}}-S\cdot\frac{{}^{n-1}d-{}^{n}d}{d}\cdot\frac{{}^{n-1}c+{}^{n}c}{2c}\rceil_{n-1}{}^{\frac{1}{2}}\mathrm{AB}.{}^{\frac{1}{2}}a\right)$$

$$=\frac{v^{\frac{1}{2}}}{a}\left(\rceil_{\mathrm{C}\underset{\mathrm{i}}{\mathrm{D}}}\mathrm{A}_{\frac{1}{2}}\mathrm{B}.a_{\frac{1}{2}}-\rceil_{\mathrm{C}\underset{\mathrm{i}}{\mathrm{D}}}{}^{\frac{1}{2}}\mathrm{AB}.{}^{\frac{1}{2}}a\right)\quad .\ .\ .\ .\ .\ .\ .\quad (2)$$

Problem X.

To determine ($\underset{\mathrm{iii}}{\mathfrak{A}}\mathbb{B}\underset{\mathrm{i}}{\mathbb{C}}\underset{\mathrm{ii}}{\mathbb{D}}$) the present value of £1 payable on the failure of the joint lives A_1, A_2, A_3, . . . provided that event shall occur after the failure of the joint lives D_1, D_2, D_3, . . . and before the failure of the joint lives B_1, B_2, B_3 . . .; and provided the joint lives C_1, C_2, C_3, . . . shall fail before the joint lives D_1, D_2, D_3, . . .

Solution.

$$\underset{\substack{\mathrm{ii.iii}\\ \mathrm{c}\\ \mathrm{ii}}}{\mathfrak{A}}\mathbb{B}\mathbb{C}\underset{\mathrm{i}}{\mathbb{D}}+\underset{\mathrm{iii}}{\mathfrak{A}}\mathbb{B}\underset{\mathrm{i}}{\mathbb{C}}\underset{\mathrm{ii}}{\mathbb{D}}=\underset{\mathrm{ii}}{\mathfrak{A}}\mathbb{B}\underset{\mathrm{i}}{\mathbb{D}}=\underset{\mathrm{i}}{\mathfrak{A}}\mathbb{B}-\underset{\mathrm{i}}{\mathfrak{A}}\mathbb{B}\mathbb{D}\text{ (Milne, XIX.)}$$

$$\therefore\ \underset{\mathrm{iii}}{\mathfrak{A}}\mathbb{B}\underset{\mathrm{i}}{\mathbb{C}}\underset{\mathrm{ii}}{\mathbb{D}}=\underset{\mathrm{i}}{\mathfrak{A}}\mathbb{B}-\underset{\mathrm{i}}{\mathfrak{A}}\mathbb{B}\mathbb{D}-\underset{\substack{\mathrm{ii.iii}\\ \mathrm{c}\\ \mathrm{ii}}}{\mathfrak{A}}\mathbb{B}\mathbb{C}\underset{\mathrm{i}}{\mathbb{D}}.$$

PROBLEM XI.

To determine $\left(\underset{\text{iii.iv}}{\mathfrak{A}}\underset{\text{ii.iii}}{\mathbb{B}}\underset{\text{i}}{\mathbb{C}}\mathbb{D}\right)$ the present value of £1 payable on the failure of the joint lives A_1, A_2, A_3, . . . provided that event shall occur after the failure of the joint lives B_1, B_2, B_3 . . .; and provided the joint lives C_1, C_2, C_3, . . . shall fail before the joint lives B_1, B_2, B_3, . . . and before the joint lives D_1, D_2, D_3, . . .

Solution.

$${}_{\rceil\underset{\text{ii.iii}}{\text{B}}\underset{\text{i}}{\text{C}}\text{D}}\text{A}+\underset{\text{iii.iv}}{\mathfrak{A}}\underset{\text{ii.iii}}{\mathbb{B}}\underset{\text{i}}{\mathbb{C}}\mathbb{D}\left(1+\frac{1}{r}\right)=\underset{\substack{\text{ii.iii}\\ \text{d}\\ \text{ii}}}{\mathfrak{B}}\mathbb{A}\underset{\text{i}}{\mathbb{C}}\mathbb{D}\left(1+\frac{1}{r}\right)$$

$$\therefore\ \underset{\text{iii.iv}}{\mathfrak{A}}\underset{\text{ii.iii}}{\mathbb{B}}\underset{\text{i}}{\mathbb{C}}\mathbb{D}=\underset{\substack{\text{ii.iii}\\ \text{d}\\ \text{ii}}}{\mathfrak{B}}\mathbb{A}\underset{\text{i}}{\mathbb{C}}\mathbb{D}-{}_{\rceil\underset{\text{ii.iii}}{\text{B}}\underset{\text{i}}{\text{C}}\text{D}}\text{A}(1-v)$$

$$=\underset{\substack{\text{ii.iii}\\ \text{d}\\ \text{ii}}}{\mathfrak{B}}\mathbb{A}\underset{\text{i}}{\mathbb{C}}\mathbb{D}-\left({}_{\rceil\underset{\text{i}}{\text{C}}\text{BD}}\text{A}-{}_{\rceil\underset{\text{i}}{\text{C}}\text{D}}\text{AB}\right)(1-v).$$

PROBLEM XII.

To determine $\left(\underset{\text{iv}}{\mathfrak{A}}\underset{\text{iii}}{\mathbb{B}}\underset{\text{ii}}{\mathbb{C}}\underset{\text{i}}{\mathbb{D}}\right)$ the present value of £1 payable on the joint lives A_1, A_2, A_3, . . . failing after the failure of the joint lives B_1, B_2, B_3, . . . provided that the joint lives C_1, C_2, C_3, . . . shall fail before the joint lives B_1, B_2, B_3, . . . and after the joint lives D_1, D_2, D_3, . . .

Solution.

$$\underset{\text{iv}}{\mathfrak{A}}\underset{\text{iii}}{\mathbb{B}}\underset{\text{ii}}{\mathbb{C}}\underset{\text{i}}{\mathbb{D}}+\underset{\text{iii.iv}}{\mathfrak{A}}\underset{\text{ii.iii}}{\mathbb{B}}\underset{\text{i}}{\mathbb{C}}\mathbb{D}=\underset{\text{iii}}{\mathfrak{A}}\underset{\text{ii}}{\mathbb{B}}\mathbb{C}=\mathfrak{A}-\underset{\text{i}}{\mathfrak{A}}\mathbb{B}-\underset{\text{i}}{\mathfrak{B}}\mathbb{A}\mathbb{C}+{}_{\rceil\underset{\text{i}}{\text{B}}\text{C}}\text{A}(1-v)$$

$$\underset{\text{iv}}{\mathfrak{A}}\underset{\text{iii}}{\mathbb{B}}\underset{\text{ii}}{\mathbb{C}}\underset{\text{i}}{\mathbb{D}}=\mathfrak{A}-\underset{\text{i}}{\mathfrak{A}}\mathbb{B}-\underset{\text{i}}{\mathfrak{B}}\mathbb{A}\mathbb{C}+{}_{\rceil\underset{\text{i}}{\text{B}}\text{C}}\text{A}(1-v)-\underset{\text{iii.iv}}{\mathfrak{A}}\underset{\text{ii.iii}}{\mathbb{B}}\underset{\text{i}}{\mathbb{C}}\mathbb{D}$$

$$=\mathfrak{A}-\underset{\text{i}}{\mathfrak{A}}\mathbb{B}-\underset{\text{i}}{\mathfrak{B}}\mathbb{A}\mathbb{C}-\underset{\substack{\text{ii iii}\\ \text{d}\\ \text{ii}}}{\mathfrak{B}}\mathbb{A}\underset{\text{i}}{\mathbb{C}}\mathbb{D}+\left({}_{\rceil\underset{\text{i}}{\text{B}}\text{C}}\text{A}+{}_{\rceil\underset{\text{i}}{\text{C}}\text{BD}}\text{A}-{}_{\rceil\underset{\text{i}}{\text{C}}\text{D}}\text{AB}\right)(1-v).$$

On a Table for the Formation of Logarithms and Anti-Logarithms to Twelve Places. By PETER GRAY, F.R.A.S., *Honorary Member of the Institute of Actuaries.*

[Read before the Institute, 27th December, 1864.]

WITH the exception of the publication, on a more limited scale (and which will be hereafter referred to), of that which is now to be developed, there exists at present no ready and practical method of forming to more than seven places the logarithms of numbers of more than seven or eight figures. The great extent of the tables requisite, if formed on the plan of our present seven-figure tables, is not only likely ever to prove a bar to their construction, but it